SOME PROPERTIES OF GROUP-THEORETICAL CATEGORIES

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Abstract. We first show that every group-theoretical category is graded by a certain double coset ring. As a consequence, we obtain a necessary and sufficient condition for a group-theoretical category to be nilpotent. We then give an explicit description of the simple objects in a group-theoretical category (following [O2]) and of the group of invertible objects of a group-theoretical category, in group-theoretical terms. Finally, under certain restrictive conditions, we describe the universal grading group of a group-theoretical category.

1. Introduction

Group-theoretical categories were introduced and studied in [ENO] and [O1]. They constitute a fundamental class of fusion categories which are defined, as the name suggests, by a certain finite group data. For example, for a finite group G its representation category $\mathrm{Rep}(G)$ is group-theoretical. As an indication of the centrality of group-theoretical categories in the theory of fusion categories we mention the following observation: all known complex semisimple Hopf algebras (as far as we know) have group-theoretical representation categories. In fact, it was asked in [ENO] whether it is true that any complex semisimple Hopf algebra is group-theoretical. It is thus highly desirable to study group-theoretical categories and understand as much as possible about them in the language of group theory.

The notion of a nilpotent fusion category was introduced and studied in [GN]. For example, it is not hard to show that if G is a finite group then $\mathrm{Rep}(G)$ is nilpotent if and only if G is nilpotent. In [DGNO] nilpotent modular categories are studied, and in particular it is discussed when they are group-theoretical. Therefore a very natural question arises: what are necessary and sufficient conditions for a group-theoretical category to be nilpotent? The answer to this question is one of the main results of this paper (see Corollary 4.3).

Other important invariants of a fusion category $\mathcal{C}$ are its pointed subcategory $\mathcal{C}_{pt}$ (the subcategory generated by the group of invertible objects in $\mathcal{C}$), its adjoint subcategory $\mathcal{C}_{ad}$ [ENO] and its universal grading group $U(\mathcal{C})$ [GN]. Descriptions of $\mathcal{C}_{pt}$ for a general group-theoretical category $\mathcal{C}$, and $\mathcal{C}_{ad}$, $U(\mathcal{C})$ for a special class of group-theoretical categories are other results of this paper (see Theorem 5.2 and Proposition 6.3).

The organization of the paper is as follows. Section 2 contains necessary preliminaries about fusion categories, module categories, and group-theoretical categories. We also recall some definitions from [GN] concerning nilpotent fusion categories and based rings. We also recall some basic definitions and results from group theory.

In Section 3 we introduce the notion of a fusion category graded by a based ring. Let H be a subgroup of a finite group G. We introduce a based ring which we call

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double coset ring arising from the set $H\backslash G/H$ of double cosets of H in G. We give a necessary and sufficient condition for the double coset ring to be nilpotent (see Proposition 3.7).

In Section 4 we first show that every group-theoretical category is graded by a certain double coset ring. As a consequence, we obtain a necessary and sufficient condition for a group-theoretical category to be nilpotent.

In Section 5 we give an explicit description of the simple objects in a group-theoretical category (following Proposition 3.2 in [O2]; see Theorem 5.1) and of the group of invertible objects of a group-theoretical category, in group-theoretical terms.

In Section 6, we describe the universal grading group of a group-theoretical category, under certain restrictive conditions.

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2. Preliminaries

2.1. **Fusion categories and their module categories.**

Throughout this paper we work over an algebraically closed field k of characteristic 0. All categories considered in this work are assumed to be k-linear and semisimple with finite dimensional Hom-spaces and finitely many isomorphism classes of simple objects. All functors are assumed to be additive and k-linear. Unless otherwise stated all cocycles appearing in this work will have coefficients in the trivial module $k^\times$.

A *fusion category* over k is a k-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects and finite dimensional Hom-spaces such that the neutral object is simple [ENO].

A fusion category is said to be *pointed* if all its simple objects are invertible. A typical example of a pointed category is Vec_G^ω - the category of finite dimensional vector spaces over k graded by the finite group G. The morphisms in this category are linear transformations that respect the grading and the associativity constraint is given by the normalized 3-cocycle ω on G.

Let $\mathcal{C} = (\mathcal{C}, \otimes, 1_\mathcal{C}, \alpha, \lambda, \rho)$ be a tensor category, where $1_\mathcal{C}$, α, λ, and ρ are the unit object, the associativity constraint, the left unit constraint, and the right unit constraint, respectively. A right *module category* over $\mathcal{C}$ (see [O1] and references therein) is a category $\mathcal{M}$ together with an exact bifunctor $\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and natural isomorphisms $\mu_{M,\,X,\,Y} : M \otimes (X \otimes Y) \to (M \otimes X) \otimes Y$, $\tau_M : M \otimes 1_\mathcal{C} \to M$, for all $M \in \mathcal{M}$, $X, Y \in \mathcal{C}$, such that the following two equations hold for all $M \in \mathcal{M}$, $X, Y, Z \in \mathcal{C}$:

$$\mu_{M\otimes X,\,Y,\,Z} \circ \mu_{M,\,X,\,Y\otimes Z} \circ (\mathrm{id}_M \otimes \alpha_{X,Y,Z}) = (\mu_{M,\,X,\,Y} \otimes \mathrm{id}_Z) \circ \mu_{M,\,X\otimes Y,\,Z},$$

$$(\tau_M \otimes \mathrm{id}_Y) \circ \mu_{M,\,1_\mathcal{C},\,Y} = \mathrm{id}_M \otimes \lambda_Y.$$

Let $(\mathcal{M}_1, \mu^1, \tau^1)$ and $(\mathcal{M}_2, \mu^2, \tau^2)$ be two right module categories over $\mathcal{C}$. A $\mathcal{C}$-*module functor* from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ together with natural isomorphisms $\gamma_{M,\,X} : F(M \otimes X) \to F(M) \otimes X$, for all $M \in \mathcal{M}_1$, $X \in \mathcal{C}$, such that

the following two equations hold for all $M \in \mathcal{M}_1$, $X, Y \in \mathcal{C}$:

$$(\gamma_{M,\,X} \otimes \mathrm{id}_Y) \circ \gamma_{M\otimes X,\,Y} \circ F(\mu^1_{M,\,X,\,Y}) = \mu^2_{F(M),\,X,\,Y} \circ \gamma_{M,X\otimes Y},$$

$$\tau^1_{F(M)} \circ \gamma_{M,\,1_{\mathcal{C}}} = F(\tau^1_M).$$

Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are *equivalent* if there exists a module functor from $\mathcal{M}_1$ to $\mathcal{M}_2$ which is an equivalence of categories. For two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over a tensor category $\mathcal{C}$ their *direct sum* is the category $\mathcal{M}_1 \oplus \mathcal{M}_2$ with the obvious module category structure. A module category is *indecomposable* if it is not equivalent to a direct sum of two non-trivial module categories.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two right module categories over a tensor category $\mathcal{C}$. Let (F^1, γ^1) and (F^2, γ^2) be module functors from $\mathcal{M}_1$ to $\mathcal{M}_2$. A *natural module transformation* from (F^1, γ^1) to (F^2, γ^2) is a natural transformation $\eta : F^1 \to F^2$ such that the following equation holds for all $M \in \mathcal{M}_1$, $X \in \mathcal{C}$:

$$(\eta_M \otimes \mathrm{id}_X) \circ \gamma^1_{M,\,X} = \gamma^2_{M,\,X} \circ \eta_{M\otimes X}.$$

Let $\mathcal{C}$ be a tensor category and let $\mathcal{M}$ be a right module category over $\mathcal{C}$. The *dual category* of $\mathcal{C}$ with respect to $\mathcal{M}$ is the category $\mathcal{C}^*_{\mathcal{M}} := Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ whose objects are $\mathcal{C}$-module functors from $\mathcal{M}$ to itself and morphisms are natural module transformations. The category $\mathcal{C}^*_{\mathcal{M}}$ is a tensor category with tensor product being composition of module functors. It is known that if $\mathcal{C}$ is a fusion category and $\mathcal{M}$ is a semisimple k-linear indecomposable module category over $\mathcal{C}$, then $\mathcal{C}^*_{\mathcal{M}}$ is a fusion category [ENO].

Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *weakly Morita equivalent* if there exists an indecomposable (semisimple k-linear) right module category $\mathcal{M}$ over $\mathcal{C}$ such that the categories $\mathcal{C}^*_{\mathcal{M}}$ and $\mathcal{D}$ are equivalent as fusion categories. It was shown by Müger [Mu] that this is indeed an equivalence relation.

Consider the fusion category Vec^{ω}_G, where G is a finite group and ω is a normalized 3-cocycle on G. Let H be a subgroup of G such that $\omega|_{H\times H\times H}$ is cohomologically trivial. Let ψ be a 2-cochain in $C^2(H, k^{\times})$ satisfying $\omega|_{H\times H\times H} = d\psi$. The twisted group algebra $k^{\psi}[H]$ is an associative unital algebra in Vec^{ω}_G. Define $\mathcal{C} = \mathcal{C}(G, \omega, H, \psi)$ to be the category of $k^{\psi}[H]$-bimodules in Vec^{ω}_G. Then $\mathcal{C}$ is a fusion category with tensor product $\otimes_{k^{\psi}[H]}$ and unit object $k^{\psi}[H]$.

Categories of the form $\mathcal{C}(G, \omega, H, \psi)$ are known as *group-theoretical* [ENO, Definition 8.40], [O2]. It is known that a fusion category $\mathcal{C}$ is group-theoretical if and only if it is weakly Morita equivalent to a pointed category with respect to some indecomposable module category [ENO, Proposition 8.42]. More precisely, $\mathcal{C}(G, \omega, H, \psi)$ is equivalent to $(\mathrm{Vec}^{\omega}_G)^*_{(H,\psi)}$.

2.2. **Nilpotent based rings and nilpotent fusion categories.**

Let $\mathbb{Z}_+$ be the semi-ring of non-negative integers. Let R be a ring with identity which is a finite rank $\mathbb{Z}$-module. A $\mathbb{Z}_+$-*basis* of R is a basis B such that for all $X, Y \in B$, $XY = \sum_{Z\in B} n^Z_{X,\,Y}\, Z$, where $n^Z_{X,\,Y} \in \mathbb{Z}_+$. An element of B will be called *basic*.

Define a non-degenerate symmetric $\mathbb{Z}$-valued inner product on R as follows. For all elements $X = \sum_{Z\in B} a_Z Z$ and $Y = \sum_{Z\in B} b_Z Z$ of R we set

$$(X, Y) = \sum_{Z\in B} a_Z b_Z. \qquad (1)$$

Definition 2.1 ([O1])**.** A *based ring* is a pair $(R,\, B)$ consisting of a ring R (with identity 1) with a $\mathbb{Z}_+$-basis B satisfying the following properties:

(1) $1 \in B$.
(2) There is an involution $X \mapsto X^*$ of B such that the induced map $X = \sum_{W\in B} a_W W \mapsto X^* = \sum_{W\in B} a_W W^*$ satisfies
$$(XY,\, Z) = (X, ZY^*) = (Y,\, X^*Z)$$
for all $X, Y, Z \in R$.

By a *based subring* of a based ring $(R,\, B)$ we will mean a based ring (S, C) where C is a subset of B and S is a subring of R.

Let us recall some definitions from [GN].

Let $R = (R,\, B)$ be a based ring and let $\mathcal{C}$ be a fusion category.

Let R_{ad} denote the based subring of R generated by all basic elements of R contained in XX^*, $X \in B$. Let $R^{(0)} := R$, $R^{(1)} := R_{ad}$, and $R^{(i)} := (R^{(i-1)})_{ad}$, for every positive integer i. Similarly, let $\mathcal{C}_{ad}$ denote the full fusion subcategory of $\mathcal{C}$ generated by all simple subobjects of $X \otimes X^*$, X a simple object of $\mathcal{C}$. Let $\mathcal{C}^{(0)} := \mathcal{C}$, $\mathcal{C}^{(1)} := \mathcal{C}_{ad}$, and $\mathcal{C}^{(i)} := (\mathcal{C}^{(i-1)})_{ad}$, for every positive integer i.

R is said to be *nilpotent* if $R^{(n)} = \mathbb{Z}1$, for some n. The smallest n for which this happens is called the *nilpotency class* of R and is denoted by $cl(R)$.

$\mathcal{C}$ is said to be *nilpotent* if $\mathcal{C}^{(n)} \cong \mathrm{Vec}$, for some n. The smallest n for which this happens is called the *nilpotency class* of $\mathcal{C}$ and is denoted by $cl(\mathcal{C})$.

Note that a fusion category is nilpotent if and only if its Grothendieck ring is nilpotent. Also note that for any finite group G, the fusion category $\mathrm{Rep}(G)$ of representations of G is nilpotent if and only if the group G is nilpotent.

Let $\mathcal{C}$ be a fusion category. We can view $\mathcal{C}$ as a $\mathcal{C}_{ad}$-bimodule category. As such, it decomposes into a direct sum of indecomposable $\mathcal{C}_{ad}$-bimodule categories: $\mathcal{C} = \oplus_{a\in A}\mathcal{C}_a$, where A is the index set. It was shown in [GN] that there is a canonical group structure on the index set A. This group is called the *universal grading group* of $\mathcal{C}$ and is denoted by $U(C)$. Every fusion category is faithfully graded (in the sense of [ENO, Definition 5.9]) by its universal grading group.

2.3. **Some definitions and results from group theory.**

The following definitions and results are contained in [R].

Let H be a subgroup of a group G. The subgroup H is said to be *subnormal* in G if there exist subgroups $H_1, \cdots H_{n-1}$ of G such that
$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = G.$$

For any non-empty subsets X and Y of G, let X^Y denote the subgroup generated by the set $\{yxy^{-1} \mid x \in X, y \in Y\}$. Define a sequence of subgroups $H^{(G,\, i)}, i = 0, 1, \ldots,$ of G by the rules
$$H^{(G,\,0)} := G \text{ and } H^{(G,\,i+1)} := H^{H^{(G,\,i)}}.$$
So we get the following sequence
$$G = H^{(G,\,0)} \trianglerighteq H^{(G,\,1)} \trianglerighteq H^{(G,\,2)} \trianglerighteq \cdots .$$
Note that $H^{(G,\,1)}$ is the normal closure of H in G. The above sequence is called the *series of successive normal closure* of H in G. It is known that H is subnormal in G if and only if $H^{(G,\,n)} = H$ for some $n \geq 0$. If H is subnormal in G, the smallest n for which $H^{(G,\,n)} = H$ is called the *defect* of H in G.

Suppose G is finite. Then it is known that G is nilpotent if and only if any subgroup of G is subnormal in G. It is also known that if H is nilpotent and is subnormal in G, then the normal closure of H in G is nilpotent. Indeed, it can be shown that if H is nilpotent and is subnormal in G, then H is contained in the Fitting subgroup $\mathrm{Fit}(G)$ of G (= the unique largest normal nilpotent subgroup of G), and hence the normal closure of H in G must be nilpotent.

3. Fusion categories graded by based rings and double coset rings

In this section we define the notion of a fusion category graded by a based ring (generalizing the notion of a fusion category graded by a finite group). We then define the double coset based ring and give a necessary and sufficient condition for it to be nilpotent.

3.1. **Fusion categories graded by based rings.**

Definition 3.1. A fusion category $\mathcal{C}$ is said to be *graded* by a based ring $(R,\, B)$ if $\mathcal{C}$ decomposes into a direct sum of full abelian subcategories $\mathcal{C} = \oplus_{X\in B}\,\mathcal{C}_X$ such that $(\mathcal{C}_X)^* = \mathcal{C}_{X^*}$ and $\mathcal{C}_X \otimes \mathcal{C}_Y \subseteq \oplus_{Z\in\{W\in B | W \text{ is contained in } XY\}}\,\mathcal{C}_Z$, for all $X, Y \in B$.

Remark 3.2. Note that the trivial component $\mathcal{C}_1$ is a fusion subcategory of $\mathcal{C}$.

Let $\mathcal{C}$ be a fusion category which is graded by a based ring $(R,\, B)$.

Definition 3.3. For any subcategory $\mathcal{D} \subseteq \mathcal{C}$, define its *support* $\mathrm{Supp}(\mathcal{D}) := \{X \in B \mid \mathcal{D} \cap \mathcal{C}_X \neq \{0\}\}$. We will say that $\mathcal{C}$ is *faithfully* graded by $(R,\, B)$ if $\mathcal{C}_X \neq \{0\}$ and $\mathrm{Supp}(\mathcal{C}_X \otimes \mathcal{C}_Y) = \{W \in B \mid W$ is contained in $XY\}$, for all $X, Y \in B$.

Remark 3.4. (i) Every fusion category is faithfully graded by its Grothendieck ring.
(ii) Every fusion category that is graded by a group G is graded by the based ring $(\mathbb{Z}G,\, G)$.

Recall that for any fusion category $\mathcal{C}$, $\mathcal{C}_{ad}$ denotes the full fusion subcategory of $\mathcal{C}$ generated by all simple subobjects of $X \otimes X^*$, X a simple object of $\mathcal{C}$; $\mathcal{C}^{(0)} = \mathcal{C}$, $\mathcal{C}^{(1)} = \mathcal{C}_{ad}$, and $\mathcal{C}^{(i)} = (\mathcal{C}^{(i-1)})_{ad}$ for every positive integer i.

Also recall that for any based ring $(R,\, B)$, R_{ad} denotes the based subring of R generated by all basic elements of R contained in XX^*, $X \in B$; $R^{(0)} = R$, $R^{(1)} = R_{ad}$, and $R^{(i)} = (R^{(i-1)})_{ad}$ for every positive integer i.

Proposition 3.5. *Let $\mathcal{C}$ be a fusion category that is faithfully graded by a based ring $R = (R,\, B)$. Then $\mathcal{C}$ is nilpotent if and only if R is nilpotent and the trivial component $\mathcal{C}_1$ is nilpotent. If $\mathcal{C}$ is nilpotent, then its nilpotency class $cl(\mathcal{C})$ satisfies the following inequality:*

$$cl(R) \leq cl(\mathcal{C}) \leq cl(R) + cl(\mathcal{C}_1).$$

Proof. Since the grading of $\mathcal{C}$ by R is faithful, we have $\mathrm{Supp}(\mathcal{C}^{(i)}) = B \cap R^{(i)}$ for any non-negative integer i. Indeed, note that even without faithfulness of the grading we have $\mathrm{Supp}(\mathcal{C}^{(i)}) \subseteq B \cap R^{(i)}$. Faithfulness of the grading implies that $B \cap R^{(i)} \subseteq \mathrm{Supp}(\mathcal{C}^{(i)})$. Now suppose that $\mathcal{C}$ is nilpotent of nilpotency class n. Then the trivial component $\mathcal{C}_1$ being a fusion subcategory of $\mathcal{C}$ is nilpotent. Also, $\mathrm{Supp}(\mathcal{C}^{(n)})$ must be equal to $\{1\}$. It follows that R must be nilpotent. Conversely, suppose that the trivial component $\mathcal{C}_1$ is nilpotent and R is nilpotent of nilpotency class n. Then $\mathcal{C}^{(n)} \subseteq \mathcal{C}_1$ and it follows that $\mathcal{C}$ must be nilpotent. The statement about nilpotency class should be evident and the proposition is proved. ∎

3.2. **The double coset ring.**

Let H be a subgroup of a finite group G. Let $\mathcal{R}(G,\, H)$ denote the free $\mathbb{Z}$-module generated by the set $\mathcal{O}$ of double cosets of H in G. For any $HxH, HyH \in \mathcal{O}$, the set $HxHyH$ is a union of double cosets. Define the product $HxH \cdot HyH$ by

$$HxH \cdot HyH := \sum_{HzH \in \mathcal{O}} N_{HxH,\, HyH}^{HzH}\, HzH,$$

where

$$N_{HxH,\, HyH}^{HzH} = \begin{cases} 1 \text{ if } HzH \subseteq HxHyH, \\ 0 \text{ otherwise.} \end{cases}$$

This multiplication rule on $\mathcal{O}$ extends, by linearity, to a multiplication rule on $\mathcal{R}(G,\, H)$. The identity element of $\mathcal{R}(G,\, H)$ is given by the trivial double coset $H = H1_G H$. There is an involution $*$ on the set $\mathcal{O}$ defined as follows. For any $HxH \in \mathcal{O}$, define $(HxH)^* := Hx^{-1}H$. It is straightforward to check that $\mathcal{R}(G,\, H)$ is a based ring.

Let $\mathcal{S}$ be a based subring of $\mathcal{R}(G,\, H)$. Define

$$\Gamma_{\mathcal{S}} := \bigcup_{X \in \mathcal{S} \cap \mathcal{O}} X.$$

Note that $\Gamma_{\mathcal{S}}$ is a subgroup of G that contains H. Also note that $\Gamma_{\mathcal{R}(G,\, H)} = G$.

Lemma 3.6. *The assignment $\mathcal{S} \mapsto \Gamma_{\mathcal{S}}$ is a bijection between the set of based subrings of the double coset ring $\mathcal{R}(G, H)$ and the set of subgroups of G containing H.*

Proof. Let K be a subgroup of G that contains H. The double coset ring $\mathcal{R}(K,\, H)$ is a based subring of $\mathcal{R}(G,\, H)$. It is evident that the assignment $K \mapsto \mathcal{R}(K,\, H)$ is inverse to the assignment defined in the statement of the lemma. ■

Proposition 3.7. *The double coset ring $\mathcal{R}(G, H)$ is nilpotent if and only if H is subnormal in G. If $\mathcal{R}(G, H)$ is nilpotent, then its nilpotency class is equal to the defect of H in G.*

Proof. Let $\mathcal{R} = \mathcal{R}(G, H)$. Observe that $\Gamma_{\mathcal{R}^{(i)}} = H^{(G,\, i)}$, for all non-negative integers i (see Subsection 2.3 for the definition of $H^{(G,\, i)}$). Note that $\mathcal{R}$ is nilpotent if and only if $H^{(G,\, n)} = H$ for some non-negative integer n. The latter condition is equivalent to the condition that H is subnormal in G. Recall that if H is subnormal in G, then the defect of H in G is defined to be the smallest non-negative integer n such that $H^{(G,\, n)} = H$. It follows that if $\mathcal{R}$ is nilpotent, then its nilpotency class is equal to the defect of H in G. ■

4. Nilpotency of a group-theoretical category

In this section we give a necessary and sufficient condition for a group-theoretical category to be nilpotent.

We start with the following theorem.

Theorem 4.1. *Let $\mathcal{C} = \mathcal{C}(G,\, \omega,\, H,\, \psi)$ be a group-theoretical category. Then $\mathcal{C}$ is faithfully graded by the double coset ring $\mathcal{R}(G,\, H)$, with the trivial component being the representation category Rep(H) of H.*

Proof. It follows from the results in [O2] that the set of isomorphism classes of simple objects in $\mathcal{C}$ are parametrized by pairs (a, ρ), where $a \in G$ is a representative of a double coset $X := HaH$ of H in G (i.e., a basic element X in $\mathcal{R}(G, H)$) and an irreducible projective representation of $H^a := H \cap aHa^{-1}$ with a certain $2-$cocycle. Moreover, the tensor product of two simple objects X, Y, corresponding to (a, ρ), (b, τ), respectively, is supported on the union of the double cosets appearing in the decomposition of XY. Therefore if we let $\mathcal{C}_X$, $X := HaH$, be the subcategory of $\mathcal{C}$ generated by all simple objects which correspond to pairs (a, ρ), we get that $\mathcal{C} = \oplus_X \mathcal{C}_X$, as required. It is clear that $\mathcal{C}_H = \mathrm{Rep}(H)$. ∎

Remark 4.2. We note that if N is the normal closure of H in G then the group ring $\mathbb{Z}[G/N]$ is a homomorphic image of $\mathcal{R}(G, H)$. Hence the group-theoretical category $\mathcal{C} = \mathcal{C}(G, \omega, H, \psi)$ is $G/N-$graded.

Corollary 4.3. *Let $\mathcal{C} = \mathcal{C}(G, \omega, H, \psi)$ be a group-theoretical category. Then $\mathcal{C}$ is nilpotent if and only if the normal closure of H in G is nilpotent. If $\mathcal{C}$ is nilpotent, then its nilpotency class $cl(\mathcal{C})$ satisfies the following inequality:*

$$cl(H) \leq cl(\mathcal{C}) \leq cl(H) + (\textit{defect of } H \textit{ in } G).$$

Proof. By Theorem 4.1 and Proposition 3.5, it follows that $\mathcal{C}$ is nilpotent if and only if the double coset ring $\mathcal{R}(G, H)$ is nilpotent and H is nilpotent. By Proposition 3.7, $\mathcal{R}(G, H)$ is nilpotent if and only if H is subnormal in G. Since G is a finite group, it follows from the remarks in Subsection 2.3 that H is nilpotent and is subnormal in G if and only if the normal closure of H in G is nilpotent. The statement about the nilpotency class of $\mathcal{C}$ follows immediately from Proposition 3.5 and Proposition 3.7. ∎

Example 4.4. Let G be a finite group and let ω be a 3-cocycle on G. It was shown in [O2] that the representation category $\mathrm{Rep}(D^\omega(G))$ of the twisted quantum double of G is equivalent to $\mathcal{C}(G \times G, \tilde{\omega}, \Delta(G), 1)$, where $\tilde{\omega}$ is a certain 3-cocycle on $G \times G$ and $\Delta(G)$ is the diagonal subgroup of G. It follows from Corollary 4.3 that $\mathrm{Rep}(D^\omega(G))$ is nilpotent if and only if G is nilpotent.

5. The pointed subcategory of a group-theoretical category

In this section we describe the simple objects in a group-theoretical category and then describe the group of invertible objects in a group-theoretical category.

5.1. **Simple objects in a group-theoretical category.**

Let $\mathcal{C} = \mathcal{C}(G, \omega, H, \psi)$ be a group-theoretical category. Let $R = \{u(X) \mid X \in H\backslash G/H\}$ be a set of representatives of double cosets of H in G. We assume that $u(H1_GH) = 1_G$. In [O2] it is explained how a simple object in $\mathcal{C}$ gives rise to a pair $(g, \overline{\rho})$, where $g \in R$ and $\overline{\rho}$ is the isomorphism class of an irreducible projective representation ρ of H^g with a certain 2-cocycle ψ^g. Let us recall this in details.

For each $g \in G$, let $H^g := H \cap gHg^{-1}$. The group H^g has a well-defined 2-cocycle ψ^g defined by

$$\psi^g(h_1, h_2) := \psi(h_1, h_2)\psi(g^{-1}h_2^{-1}g, g^{-1}h_1^{-1}g)\frac{\omega(h_1, h_2, g)\omega(h_1, h_2g, g^{-1}h_2^{-1}g)}{\omega(h_1h_2g, g^{-1}h_2^{-1}g, g^{-1}h_1^{-1}g)}.$$

Let $B = \oplus_{g\in G} B_g$ be an object in $\mathcal{C}$. So B is equipped with isomorphisms $l_{h,g} : B_g \xrightarrow{\sim} B_{hg}$ and $r_{g,h} : B_g \xrightarrow{\sim} B_{gh}$, $g \in G, h \in H$. These isomorphisms satisfy

the following identities:

$$\omega(h_1,\, h_2,\, g)\psi(h_1,\, h_2)l_{h_1h_2,\, g} = l_{h_1,\, h_2g} \circ l_{h_2,\, g},$$

$$\psi(h_1,\, h_2)r_{g,\, h_1h_2} = \omega(g,\, h_1,\, h_2)r_{gh_1,\, h_2} \circ r_{g,\, h_1}$$

and

$$l_{h_1,\, gh_2} \circ r_{g,\, h_2} = \omega(h_1,\, g,\, h_2)r_{h_1g,\, h_2} \circ l_{h_1,\, g}.$$

The above three identities say that B is a left $k^\psi[H]$-module, B is a right $k^\psi[H]$-module, and that the left and right module structures on B commute, respectively. It is clear that B is a direct sum of subbimodules supported on individual double cosets of H in G. Suppose B contains a subbimodule that is supported on a double coset represented by g. Then one get a projective representation $\rho : H^g \to GL(V)$ with 2-cocycle ψ^g defined as follows. Let $V := B_g$ and

$$\rho(h) := r_{hg,\, g^{-1}h^{-1}g} \circ l_{h,\, g}, \qquad h \in H^g. \tag{2}$$

The following theorem, stated in [O2], asserts that the above correspondence gives a bijection between isomorphism classes of simple objects in $\mathcal{C}$ and isomorphism classes of pairs $(g,\, \rho)$. We shall give an alternative proof of the inverse correspondence by a direct computation.

Theorem 5.1. *The above correspondence defines a bijection between isomorphism classes of simple objects in $\mathcal{C}$ and isomorphism classes of pairs $(g,\, \rho)$, where $g \in R$ and ρ is an irreducible projective representation of H^g with $2-$cocycle ψ^g.*

Proof. Given a pair $(g,\, \rho)$, where $g \in R$ and $\rho : H^g \to GL(V)$ is an irreducible projective representation with 2-cocycle ψ^g, we assign an object B in $\mathcal{C}$ as follows. Let T be a set of representatives of H/H^g. We assume that $1 \in T$. Let $B := \oplus_{t\in T, k\in H} B_{tgk}$, where each component is equal to V as a vector space. The right and left module structures r and l, respectively, on B are defined as follows.

$$r_{tgk,\, h} : B_{tgk} \xrightarrow{\sim} B_{tgkh}, v \mapsto \psi(k,\, h)\omega(tg,\, k,\, h)^{-1}v. \tag{3}$$

$$\begin{aligned} l_{h,\, tgk} : B_{tgk} \xrightarrow{\sim} B_{sg(g^{-1}pg)h}, v \mapsto &\frac{\psi(h,\, t)}{\psi(s,\, p)\psi(g^{-1}p^{-1}g,\, g^{-1}pgk)} \\ \times &\frac{\omega(h,\, tg,\, k)\omega(s,\, g,\, g^{-1}pg)\omega(h,\, t,\, g)}{\omega(s,\, p,\, g)} \\ \times &\frac{\omega(g,\, g^{-1}pg,\, g^{-1}p^{-1}g)\omega(g^{-1}pg,\, g^{-1}p^{-1}g,\, g^{-1}pgk)}{\omega(sg,\, g^{-1}pg,\, k)}\rho(p)(v), \end{aligned} \tag{4}$$

where $s \in T$ and $p \in H^g$ are uniquely determined by the equation $ht = sp$. It is now straightforward to check that B is simple, and that the two correspondences are inverse to each other. ∎

5.2. **The group of invertible objects in a group-theoretical category.**

For any $g \in N_G(H)$ and $f \in C^n(H,\, k^\times)$, define ${}^g f \in C^n(H,\, k^\times)$ by

$${}^g f(h_1, \cdots, h_n) := f(g^{-1}h_1g, \cdots, g^{-1}h_ng).$$

Pick any $g_1, g_2 \in N_G(H)$ and let $g_3 = g_1 g_2 k, k \in H$. Define

(5)
$$\beta(g_1,\, g_2) : H \to k^\times, h \mapsto \frac{\psi(g_2^{-1} g_1^{-1} h g_1 g_2 k,\, g_3^{-1} h^{-1} g_3)}{\psi(g_1^{-1} h^{-1} g_1,\, g_1^{-1} h g_1)\psi(g_2^{-1} g_1^{-1} h^{-1} g_1 g_2,\, g_2^{-1} g_1^{-1} h g_1 g_2 k)}$$
$$\times \frac{\omega(g_1^{-1} h g_1,\, g_1^{-1} h^{-1} g_1,\, g_1^{-1} h g_1)\omega(g_1,\, g_1^{-1} h g_1,\, g_1^{-1} h^{-1} g_1)\omega(g_1^{-1} h g_1,\, g_2,\, k)}{\omega(g_2,\, g_2^{-1} g_1^{-1} h g_1 g_2,\, k)}$$
$$\times \frac{\omega(g_2^{-1} g_1^{-1} h g_1 g_2,\, g_2^{-1} g_1^{-1} h^{-1} g_1 g_2, g_2^{-1} g_1^{-1} h g_1 g_2 k)\omega(g_2,\, g_2^{-1} g_1^{-1} h g_1 g_2,\, g_2^{-1} g_1^{-1} h^{-1} g_1 g_2)}{\omega(g_2,\, g_2^{-1} g_1^{-1} h g_1 g_2 k,\, g_3^{-1} h^{-1} g_3)}.$$

It is straightforward (but tedious) to verify that

$$\psi^{g_3} = d(\beta(g_1,\, g_2))\, \psi^{g_1}\, ({}^{g_1}(\psi^{g_2})). \tag{6}$$

Let $K := \{g \in R \mid g \in N_G(H) \text{ and } \psi^g \text{ is cohomologically trivial}\}$. For any $g_1, g_2 \in K$, define $g_1 \cdot g_2 := u(g_1 g_2)$. It follows from (6) that with this product rule K is a group that is isomorphic to a subgroup of $N_G(H)/H$.

For each $g \in K$, fix $\eta_g : H \to k^\times$ such that $d\eta_g = \psi^g$. We take $\eta_1 := \beta(1,\, 1)^{-1}$. For any $g_1, g_2 \in K$, define

$$\nu(g_1,\, g_2) := \frac{\eta_{g_1}({}^{g_1}\eta_{g_2})}{\eta_{g_1 \cdot g_2}} \beta(g_1,\, g_2). \tag{7}$$

Let $\widehat{H} := \mathrm{Hom}(H,\, k^\times)$ and define a group $K \ltimes_\nu \widehat{H}$ as follows. As a set $K \ltimes_\nu \widehat{H} = K \times \widehat{H}$ and for any $(g_1,\, \rho_1), (g_2,\, \rho_2) \in K \ltimes_\nu \widehat{H}$, define

$$(g_1,\, \rho_1) \cdot (g_2,\, \rho_2) = (g_1 \cdot g_2,\, \nu(g_1,\, g_2)\rho_1({}^{g_1}\rho_2)).$$

Theorem 5.2. *The group $G(\mathcal{C})$ of isomorphism classes of invertible objects of $\mathcal{C}$ is isomorphic to the group $K \ltimes_\nu \widehat{H}$ constructed above.*

Proof. By Theorem 5.1, $G(\mathcal{C})$ is in bijection with the set

$$L = \{(g,\, \rho) \mid g \in K, \rho : H \to k^\times \text{ such that } d\rho = \psi^g\}.$$

The set L becomes a group with product

$$(g_1,\, \rho_1) \cdot (g_2,\, \rho_2) = (g_1 \cdot g_2,\, \beta(g_1,\, g_2)\rho_1({}^{g_1}\rho_2)).$$

The identity element of L is $(1,\, \beta(1,\, 1)^{-1})$. Let B, B' be objects in $\mathcal{C}$ corresponding to $(g_1,\, \rho_1), (g_2,\, \rho_2) \in L$, respectively. So $B = \oplus_{h \in H} k_{g_1 h}$ and $B' = \oplus_{h \in H} k_{g_2 h}$, where each component is equal to the ground field k. The right and left module structures on B, B' are defined via (3) and (4). Let $A := k^\psi[H]$. We have $B \otimes_A B' = (k_{g_1} A) \otimes_A (\oplus_{h \in H} k_{g_2 h}) = k_{g_1} \otimes (\oplus_{h \in H} k_{g_2 h})$. Taking into account (3) and (4) we calculate that the projective representation (defined in (2)) $\rho : H \to k^\times$ with 2-cocycle ψ^{g_3}, corresponding to $B \otimes_A B'$, where $g_3 = g_1 \cdot g_2$, is given by $\beta(g_1,\, g_2)\rho_1({}^{g_1}\rho_2)$. So $G(\mathcal{C})$ is isomorphic to the group L. The map $L \to K \ltimes_\nu \widehat{H} : (g,\, \rho) \mapsto (g,\, \eta_g^{-1}\rho)$ establishes the desired isomorphism and the theorem is proved. ■

6. The universal grading group of certain group-theoretical categories

Recall that every fusion category $\mathcal{C}$ is faithfully graded by its universal grading group $U(\mathcal{C})$: $\mathcal{C} = \oplus_{x \in U(\mathcal{C})} \mathcal{C}_x$. In this section we describe $U(\mathcal{C})$ for certain group-theoretical categories.

Lemma 6.1. *Let $\mathcal{D}$ be a fusion category and let $\mathcal{E}$ be a fusion subcategory of $\mathcal{D}$. The map $U(\mathcal{E}) \to U(\mathcal{D})$ defined by the rule $x \mapsto y$ if and only if $\mathcal{E}_x \subseteq \mathcal{D}_y \cap \mathcal{E}$ is a homomorphism. This homomorphism is injective if and only if $\mathcal{D}_{ad} \cap \mathcal{E} = \mathcal{E}_{ad}$.*

Proof. We have universal gradings: $\mathcal{D} = \oplus_{y\in U(\mathcal{D})}\mathcal{D}_y$ and $\mathcal{E} = \oplus_{x\in U(\mathcal{E})}\mathcal{E}_x$. From the former grading we obtain $\mathcal{E} = \mathcal{D} \cap \mathcal{E} = \oplus_{y\in U(\mathcal{D})}(D_y \cap \mathcal{E})$. Note that this grading need not be faithful. Since $\mathcal{E}_{ad} \subseteq \mathcal{D}_{ad} \cap \mathcal{E}$, each component $D_y \cap \mathcal{E}$ is a $\mathcal{E}_{ad}$-submodule category of $\mathcal{E}$. So, for every $x \in U(\mathcal{E})$ there is a unique $y \in U(\mathcal{D})$ such that $\mathcal{E}_x \subseteq \mathcal{D}_y$. This gives rise to a homomorphism $U(\mathcal{E}) \to U(\mathcal{D})$. It is evident that this homomorphism is injective if and only if $\mathcal{D}_{ad} \cap \mathcal{E} = \mathcal{E}_{ad}$. ■

Lemma 6.2. *The universal grading group $U(Rep(K))$ of the representation category of a finite group K is isomorphic to the center $Z(K)$ of K.*

Proof. This is a special case of Theorem 3.8 in [GN] (H being the group algebra of K). ■

Proposition 6.3. *Let $\mathcal{C} = \mathcal{C}(G, 1, H, 1)$. Suppose H is normal in G. Then there is a split exact sequence $1 \to Z(H) \to U(\mathcal{C}) \to G/H \to 1$. Therefore, $U(\mathcal{C})$ is isomorphic to the semi-direct product $G/H \ltimes Z(H)$.*

Proof. By Theorem 4.1, we have a grading of $\mathcal{C}$ by the group G/H: $\mathcal{C} = \oplus_{x\in G/H}\mathcal{C}^x$, where $\mathcal{C}^x$ is the full abelian subcategory of $\mathcal{C}$ consisting of objects supported on the coset x. Let $\mathcal{E} := \mathcal{C}^1$. We will first show that $\mathcal{C}_{ad} = \mathcal{E}_{ad}$. Let R be a set representatives of cosets of H in G. Recall that simple objects of $\mathcal{C}$ correspond to pairs (a, ρ), where $a \in R$ and ρ is an irreducible representation of H. Let B be the object in $\mathcal{C}$ corresponding to (a, ρ) defined via (3) and (4). The dual object B^* corresponds to the pair $(b, ({}^b\rho)^*)$, where $b \in R$ is the representative of the coset $a^{-1}H$. The representation (defined in (2)) corresponding to $B \otimes_{k[H]} B^*$ is given by $\rho \otimes {}^a(({}^b\rho)^*) \cong \rho \otimes \rho^*$. This establishes the equality $\mathcal{C}_{ad} = \mathcal{E}_{ad}$.

By Theorem 4.1, $\mathcal{E} \cong \mathrm{Rep}(H)$ and Lemma 6.2 implies that $U(\mathcal{E}) \cong Z(H)$. By Lemma 6.1, we get an injective homomorphism $i : Z(H) \to U(\mathcal{C})$. From [GN, Corollary 3.7] we get a surjective homomorphism $p : U(\mathcal{C}) \to G/H$ which is defined as follows. Note that $\mathcal{E}$ contains $\mathcal{C}_{ad}$. Therefore, each $\mathcal{C}^x$ is a $\mathcal{C}_{ad}$-submodule category of $\mathcal{C}$. So, for every $y \in U(\mathcal{C})$ there is a unique $p(y) \in G/H$ such that the component $\mathcal{C}_y$ of the universal grading $\mathcal{C} = \oplus_{z\in U(\mathcal{C})}\mathcal{C}_z$ is contained in $\mathcal{C}^{p(y)}$.

We claim that the sequence $1 \to Z(H) \xrightarrow{i} U(\mathcal{C}) \xrightarrow{p} G/H \to 1$ is exact. We have $\mathcal{C}_{ad} = \mathcal{E}_{ad} \cong \mathrm{Rep}(H)_{ad} \cong \mathrm{Rep}(H/Z(H))$. By [ENO, Proposition 8.20], it follows that $|U(\mathcal{C})| = |Z(H)|\frac{|G|}{|H|}$ and therefore $|\mathrm{Ker}\ p| = |Z(H)|$. So, it suffices to show that $\mathrm{Ker}\ p \subseteq \mathrm{Im}\ i$. We have $\mathrm{Ker}\ p = \{y \in U(\mathcal{C}) \mid \mathcal{C}_y \subseteq \mathcal{E}\}$. Pick any $y \in \mathrm{Ker}\ p$ and let $K := \{y \in U(\mathcal{C}) \mid \mathcal{C}_y \cap \mathcal{E} \neq \{0\}\}$. Then $\mathcal{E} = \oplus_{k\in K}(\mathcal{C}_k \cap \mathcal{E})$ is a faithful grading of $\mathcal{E}$. Note that $y \in K$. By [GN, Corollary 3.7], there exists $z \in U(\mathcal{E})$ such that $\mathcal{E}_z \subseteq \mathcal{C}_y$, i.e., $y \in \mathrm{Im}\ i$. This establishes the exactness of the aforementioned sequence.

Finally, we show that the aforementioned sequence splits. Let $\mathcal{D}$ be the full fusion subcategory of $\mathcal{C}$ generated by simple objects in $\mathcal{C}$ corresponding to pairs (a, ρ_0), where $a \in R$ and ρ_0 is the trivial representation of H. Note that $\mathcal{D} \cong \mathrm{Vec}_{G/H}$ and $U(\mathcal{D}) \cong G/H$. Also note that $\mathcal{C}_{ad} \cap \mathcal{D} = D_{ad} \cong \mathrm{Vec}$. So, by Lemma 6.1 we obtain an injection $j : G/H \to U(\mathcal{C})$. We claim that $p \circ j = \mathrm{id}_{G/H}$. Pick any $x \in G/H$ and let $j(x) = y$, i.e., $\mathcal{D}_x \subseteq \mathcal{C}_y$. We have $\mathcal{C}_y \subseteq \mathcal{C}^{p(y)}$ which implies that $D_x \subseteq \mathcal{C}^{p(y)}$. It follows that $p(y) = x$ and the proposition is proved. ■

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